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Born expansions for Coulomb-type interactions

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Abstract. We prove lower bounds on the radius of convergence of various Born expansions associated with partial wave Schrödinger operators h_l involving Coulomb plus short-range potentials. Our estimates, clearly indicating the high-energy character of these expansions, also exhibit the connection between the increase of the radii of convergence with increasing energy and the behaviour of the short-range potential at the origin. In the case of a repulsive Coulomb potential, we prove bounds on the number of eigenvalues of h_l and discuss the relation between the absence of bound states of h_l and the convergence of Born expansions.

1. Introduction

In this paper we concentrate on the Born expansions associated with Hamiltonians including long-range potentials like the Coulomb potential. In particular, we consider Schrödinger operators h_l in $L^2(0, \infty)$ which are distinguished self-adjoint realisations of differential expressions of the type

$$d_l = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)+\alpha^2-\frac{1}{4}}{r^2} + \frac{\gamma}{r} + gV(r), \qquad r>0,\ l\in N_0,\ \alpha>0,\ \gamma\in R,\ g\in C, \tag{1.1}$$

where the short-range potential $V(r)$ is a real-valued locally integrable function on $(0, \infty)$ satisfying appropriate integrability conditions (cf § 2). In the spherically symmetric case considered here, essentially the following two types of Born expansions (Taylor series in the coupling constant g) exist:

(i) the Born expansions for $\tilde{F}_l(k, g, r)$ and $\tan \delta_l(k, g)$ which are obtained by iterating

$$\tilde{F}_l(k, g, r) = F_l^{(0)}(k, r) - g\int_0^\infty \mathrm{d}r'\, \tilde{g}_l^{(0)}(k, r, r')V(r')\tilde{F}_l(k, g, r'), \qquad k\geq 0, \tag{1.2}$$

and inserting it into

$$\tan \delta_l(k, g) = -\frac{\pi g k^{2l+1}}{2^{2l+2}\Gamma^2(l+3/2)}\int_0^\infty \mathrm{d}r\, V(r)F_l^{(0)}(k, r)\tilde{F}_l(k, g, r), \qquad k>0; \tag{1.3}$$

(ii) the Born expansions for $\tilde{\tilde{F}}_l(k, g, r)$ and $\exp[2\mathrm{i}\delta_l(k, g)]$, using

$$\tilde{\tilde{F}}_l(k, g, r) = F_l^{(0)}(k, r) - g\int_0^\infty \mathrm{d}r'\, \tilde{\tilde{g}}_l^{(0)}(k, r, r')V(r')\tilde{\tilde{F}}_l(k, g, r'), \qquad k\geq 0, \tag{1.4}$$

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and

$$\exp[2i\delta_l(k, g)] = 1 - \frac{i\pi k^{2l+1} g}{2^{2l+1}\Gamma^2(l+3/2)} \int_0^\infty dr\, V(r) F_l^{(0)}(k, r) \tilde{\tilde{F}}_l(k, g, r), \qquad k > 0. \quad (1.5)$$

Here

$$F_l^{(0)}(k, r) = \Gamma(l+3/2)(k/2)^{-l-1/2} r^{1/2} J_{l+1/2}(kr)$$

and the Green functions $\tilde{g}_l^{(0)}(k, r, r')$ and $\tilde{\tilde{g}}_l^{(0)}(k, r, r')$ follow from (3.16) and (3.41) in the limit $\gamma \to 0$.

Jost and Pais (1951), considering case (ii), proved a sufficient condition for convergence at all energies. Subsequently Kohn (1954) investigated cases (i) and (ii). He proved sufficient conditions for convergence of the Born expansions involved, and also derived lower and upper bounds on the radius of convergence for various energy regions. Later on, Zemach and Klein (1958) rigorously demonstrated (for a restricted class of potentials) the high-energy character of these expansions by proving convergence of the Born series for any fixed coupling constant g for sufficiently high energies. Their analysis has been generalised by Aaron and Klein (1960) to arbitrary space dimensions. The connection between the absence of bound states for the potential $-|gV(|x|)|$ and the convergence of all Born expansions for all energies can be found in Davies (1959/60), Meetz (1962), Huby (1963) and Bushell (1972). A discussion of case (ii), including lower bounds on the radius of convergence, appeared in Scadron *et al* (1964). Finally, we mention a generalisation of the results of Zemach and Klein (1958), due to Faris (1971) who used time decay estimates in the framework of time-dependent scattering theory.

In (1.1) we introduced a terminology especially convenient to describe radial decompositions of Hamiltonians in $L^2(R^3)$. Nevertheless, because of the arbitrariness of $\alpha \in R$, Hamiltonians in $L^2(R^n)$ of the type

$$-\Delta + \frac{\beta - (n-2)^2}{4|x|^2} + \frac{\gamma}{|x|} + gV(|x|), \qquad x \in R^n - \{0\},\ n \geq 2,\ \beta > 0,\ \gamma \in R,\ g \in C,$$

are of course included in our discussion. In treating Schrödinger operators of the type (1.1) we split up h_l into two parts, $h_l = h_l^{(0)} + gV$ (in the sense of quadratic forms), and interpret the exactly solvable operator $h_l^{(0)} = -d^2/dr^2 + [l(l+1) + \alpha^2 - \frac{1}{4}]/r^2 + \gamma/r$ as the 'unperturbed' Hamiltonian. This enables one to generalise almost all classical results of the short-range case $\gamma = 0$ by considering $h_l^{(0)}$ instead of $-d^2/dr^2 + l(l+1)/r^2$.

In § 2 we describe the spectral properties of h_l for real g. Besides a discussion of the continuous spectrum of h_l (proposition 3), we also deal with its point spectrum and prove bounds on the number of bound states of h_l in the repulsive case $\gamma \geq 0$ (propositions 1, 2). One of these bounds (proposition 2) is a straightforward generalisation of the corresponding short-range ($\gamma = 0$) result due to Bargmann (1952). In the first part of § 3 we study case (i), the Born expansions for $\tilde{F}_l(k, \gamma, g, r)$ and $\tan[\delta_l(k, \gamma, g) - \delta_l^{(0)}(k, \gamma)]$ ($\delta_l^{(0)}(k, \gamma)$ denote the phase shifts corresponding to $g = 0$) associated with our Hamiltonian h_l (i.e. in the presence of an additional long-range potential of the type $(\alpha^2 - \frac{1}{4})/r^2 + \gamma/2$). Following Kohn (1954), we derive lower bounds on the radius of convergence of the Born series in case (i) for several energy regions (propositions 4, 5). Proposition 6, which is new also in the short-range case $\gamma = 0$, connects the behaviour of $V(r)$ in a neighbourhood of the origin $r = 0$ with the increase of the radius of convergence in the high-energy limit. Incidentally, the estimate of proposition 6 proves, for sufficiently high energies, the convergence of the Born series for any fixed coupling strength g. In the special case, where $V(r)$ is integrable on $(0, \infty)$, it is also possible to prove in the high-energy limit an asymptotic expression for the radius of convergence identical to that obtained by Kohn (1954) for the short-range case $\gamma = 0$. The second part of § 3 deals with analogous results for the case (ii), i.e. for the Born expansions of $\tilde{F}_l(k, \gamma, g, r)$ and $\exp\{2i[\delta_l(k, \gamma, g) - \delta_l^{(0)}(k, \gamma)]\}$.

2. Spectral properties of h_l

This section is dedicated to a detailed description of the spectra of the Hamiltonians h_l and of the regular and irregular solutions associated with them.

Let $V(r)$ be real-valued, $V(r) \in L^1_{\text{loc}}(0, \infty)$ and

$$\int_0^R \mathrm{d}r\, r\,|V(r)| < \infty, \qquad \int_R^\infty \mathrm{d}r\,|V(r)| < \infty \qquad \text{for some } R > 0. \tag{2.1}$$

In the Hilbert space $L^2(0, \infty)$ we introduce the operator h_l by (cf (1.1))

$$\begin{aligned} &(h_l f)(r) = (d_l f)(r), \\ &D(h_l) = \{f | f' \in A_{\text{loc}}(0, \infty); f(0_+) = 0; f, f', d_l f \in L^2(0, \infty)\}. \end{aligned} \tag{2.2}$$

Here $A_{\text{loc}}(0, \infty)$ denotes the set of locally absolutely continuous functions on $(0, \infty)$. If $g \in R$, then h_l is self-adjoint (cf the discussion in Gesztesy *et al* 1980). In the special case $g = 0$ we denote the resulting 'unperturbed' operator by $h_l^{(0)}$.

Next we introduce for $E \leqslant 0$, $\gamma \geqslant 0$ and $E \geqslant 0$, $\gamma \in R$ regular and irregular solutions of the equation (cf (1.1))

$$(d_l - E)\psi(r) = 0, \qquad r > 0, \quad l \in N_0. \tag{2.3}$$

Regular solution†

$$F_l(E, \gamma, g, r) = F_l^{(0)}(E, \gamma, r) - g \int_0^r \mathrm{d}r'\, g_l^{(0)}(E, \gamma, r, r') V(r') F_l(E, \gamma, g, r'), \tag{2.4}$$

Irregular solution

$$G_l(E, \gamma, g, r) = G_l^{(0)}(E, \gamma, r) + g \int_r^\infty \mathrm{d}r'\, g_l^{(0)}(E, \gamma, r, r') V(r') G_l(E, \gamma, g, r'). \tag{2.5}$$

Here $F_l^{(0)}$ and $G_l^{(0)}$ are regular and irregular solutions for the unperturbed Hamiltonian $h_l^{(0)}$ in the case $g = 0$ (cf the Appendix). They are given by the following expressions‡.

$E \leqslant 0$, $\gamma \geqslant 0$:

$$\begin{aligned} &F_l^{(0)}(E, \gamma, r) = r^\lambda \exp(-\sqrt{-E}r)\, {}_1F_1(\lambda + \gamma/2\sqrt{-E}; 2\lambda; 2\sqrt{-E}r), \\ &G_l^{(0)}(E, \gamma, r) = \Gamma(2\lambda)^{-1}\Gamma(\lambda + \gamma/2\sqrt{-E}) \\ &\qquad \times (-4E)^{\lambda - 1/2} r^\lambda \exp(-\sqrt{-E}r) U(\lambda + \gamma/2\sqrt{-E}; 2\lambda; 2\sqrt{-E}r), \end{aligned} \tag{2.6}$$

† We always keep $\alpha > 0$ and suppress the α dependence of F_l, G_l, $g_l^{(0)}$ etc.

‡ In equation (2.6) ${}_1F_1(a; b; z)$ and $U(a; b; z)$ denote the regular and irregular confluent hypergeometric functions respectively (Abramowitz and Stegun 1972). Note that, in contrast to Gesztesy *et al* (1980), we use different definitions for $F_l^{(0)}$ and $G_l^{(0)}$ so that the limits $E \to 0$ and $\gamma \to 0$ may be performed successively (cf the Appendix).

where

$$\lambda = \tfrac{1}{2} + (l^2 + l + \alpha^2)^{1/2}.$$

$E \geqslant 0,\ \gamma \in R$:

$$F_l^{(0)}(E, \gamma, r) = r^\lambda \exp(-\mathrm{i}\sqrt{E}r)\,{}_1F_1(\lambda - \mathrm{i}\gamma/2\sqrt{E};\ 2\lambda\ ;\ 2\mathrm{i}\sqrt{E}r),$$

$$G_l^{(0)}(E, \gamma, r) = \Gamma(2\lambda)^{-1}\Gamma(\lambda - \mathrm{i}\gamma/2\sqrt{E})(4E)^{\lambda - 1/2}\exp[\mathrm{i}\pi(\lambda - 1/2)]r^\lambda \times \exp(-\mathrm{i}\sqrt{E}r)U(\lambda - \mathrm{i}\gamma/2\sqrt{E};\ 2\lambda\ ;\ 2\mathrm{i}\sqrt{E}r). \tag{2.7}$$

The unperturbed Green function $g_l^{(0)}(E, \gamma, r, r')$ is defined through

$$g_l^{(0)}(E, \gamma, r, r') = F_l^{(0)}(E, \gamma, r')G_l^{(0)}(E, \gamma, r) - F_l^{(0)}(E, \gamma, r)G_l^{(0)}(E, \gamma, r'). \tag{2.8}$$

It is simple to rewrite (2.4) and (2.5) to obtain

$$F_l(E, \gamma, g, r) = \tilde{\mathscr{F}}_l(E, \gamma, g)F_l^{(0)}(E, \gamma, r) - g\int_0^\infty \mathrm{d}r'\ \hat{g}_l^{(0)}(E, \gamma, r, r')V(r')F_l(E, \gamma, g, r') \tag{2.9}$$

and

$$G_l(E, \gamma, g, r) = \tilde{\mathscr{F}}_l(E, \gamma, g)G_l^{(0)}(E, \gamma, r) - g\int_0^\infty \mathrm{d}r'\ \hat{g}_l^{(0)}(E, \gamma, r, r')V(r')G_l(E, \gamma, g, r') \tag{2.10}$$

where we have abbreviated†

$$\tilde{\mathscr{F}}_l(E, \gamma, g) = W(G_l, F_l) = 1 + g\int_0^\infty \mathrm{d}r\ V(r)G_l^{(0)}(E, \gamma, r)F_l(E, \gamma, g, r) \tag{2.11}$$

and‡

$$\hat{g}_l^{(0)}(E, \gamma, r, r') = \begin{cases} F_l^{(0)}(E, r')G_l^{(0)}(E, r), & r' \leqslant r, \\ F_l^{(0)}(E, r)G_l^{(0)}(E, r'), & r' \geqslant r. \end{cases} \tag{2.12}$$

Equations (2.4) and (2.5) for $E \leqslant 0$, $\gamma \geqslant 0$ and $E \geqslant 0$, $\gamma \in R$ are uniquely solved by iteration provided $V(r)$ fulfils the conditions

$$\int_0^\infty \mathrm{d}r \frac{r}{1+r}|V(r)| < \infty \qquad \text{if } E < 0,\ \gamma \geqslant 0 \text{ and } E > 0,\ \gamma \in R, \tag{2.13}$$

$$\int_0^\infty \mathrm{d}r \frac{r}{1+r^{1/2}}|V(r)| < \infty \qquad \text{if } E = 0,\ \gamma \neq 0, \tag{2.14}$$

$$\int_0^\infty \mathrm{d}r\, r|V(r)| < \infty \qquad \text{if } E = \gamma = 0. \tag{2.15}$$

For a survey of estimates for F_l and G_l we refer to the Appendix.

Now we turn to the point spectrum of h_l for g real. For $\gamma < 0$ there are obviously infinitely many bound states, hence we restrict our attention to the case $\gamma \geqslant 0$.

Let us denote by $n_l(gV; \gamma; E \leqslant E_0)$, $g \geqslant 0$, the number of bound states of h_l with bound state energy less than or equal to E_0. We also introduce $V_\pm(r) =$

† $W(G, F) = G\partial F/\partial r - F\partial G/\partial r$ denotes the Wronskian of G and F.

‡ From now on we suppress the γ and g dependence whenever possible.

$[|V(r)| \pm V(r)]/2$ and exclude the trivial case where $V_-(r)=0$ ae on $(0,\infty)$. As a first result we state the following proposition.

Proposition 1. Let $E_0<0$, $\gamma \geq 0$, $g>0$ and suppose $\int_0^\infty \mathrm{d}r\,[r/(1+r)]|V(r)|<\infty$. Then

$$n_l(gV;\gamma;E\leq E_0)<\Gamma(2\lambda)^{-1}\Gamma(\lambda+\gamma/2\sqrt{-E_0})(-4E_0)^{\lambda-1/2}g\int_0^\infty \mathrm{d}r\,r^{2\lambda}\exp(-2\sqrt{-E_0}r)$$
$$\times\,{}_1F_1(\lambda+\gamma/2\sqrt{-E_0};2\lambda;2\sqrt{-E_0}r)U(\lambda+\gamma/2\sqrt{-E_0};2\lambda;2\sqrt{-E_0}r)V_-(r). \tag{2.16}$$

Proof. Let $V\leq 0$. The infinitesimal form-boundedness of V relative to $h_l^{(0)}$ implies continuity and monotonic decrease of the eigenvalues with respect to the coupling constant g (Simon 1971, Reed and Simon 1978). Thus (following Schwinger (1961) and Birman (1966)) $n_l(gV;\gamma;E\leq E_0)$ is the number of positive $\kappa\leq 1$ for which

$$[(h_l^{(0)}+\kappa(gV))\psi](r)=E_0\psi(r) \tag{2.17}$$

has a solution $\psi\in D(h_l^{(0)}+\kappa gV)$. (Here $h_l^{(0)}+\kappa gV$ denotes the form sum of $h_l^{(0)}$ and κgV.) This implies $\tilde{\mathscr{F}}_l(E_0,g)=0$ and thus

$$\phi(r)=\kappa\int_0^\infty \mathrm{d}r'\,g|V(r)|^{1/2}\hat{g}_l^{(0)}(E_0,r,r')|V(r')|^{1/2}\phi(r'),$$
$$\phi(r)=|V(r)|^{1/2}\psi(r), \tag{2.18}$$

which is equivalent to

$$\phi=\kappa g|V|^{1/2}(h_l^{(0)}-E_0)^{-1}|V|^{1/2}\phi. \tag{2.18'}$$

Under the hypothesis on $V(r)$, $|V|^{1/2}(h_l^{(0)}-E_0)^{-1}|V|^{1/2}$ is trace class (Reed and Simon 1979) and we finally obtain

$$\|g|V|^{1/2}(h_l^{(0)}-E_0)^{-1}|V|^{1/2}\|_1=g\int_0^\infty \mathrm{d}r\,\hat{g}_l^{(0)}(E_0,r,r)|V(r)|$$
$$=\sum_{n=1}^{\infty}\kappa_n^{-1}>\sum_{\{n|\kappa_n\leq 1\}}\kappa_n^{-1}\geq n(gV;\gamma;E\leq E_0), \tag{2.19}$$

where κ_n^{-1}, $n=1,2,3,\ldots$ are the eigenvalues of $g|V|^{1/2}(h_l^{(0)}-E_0)^{-1}|V|^{1/2}$. If V does not obey $V\leq 0$, we use the min–max principle (Reed and Simon 1978, Thirring 1979) to conclude that

$$n_l(gV;\gamma;E\leq E_0)\leq n_l(-gV_-;\gamma;E\leq E_0),$$

completing the proof.

In order to compare with the short-range case $\gamma=0$, we present a corollary.

Corollary 1. Let $E_0<0$, $\gamma=0$, $g>0$ and assume $\int_0^\infty \mathrm{d}r\,[r/(1+r)]|V(r)|<\infty$. Then†

$$n_l(gV;0;E\leq E_0)<g\int_0^\infty \mathrm{d}r\,rI_{\lambda-1/2}(\sqrt{-E_0}r)K_{\lambda-1/2}(\sqrt{-E_0}r)V_-(r). \tag{2.20}$$

† Here $I_\beta(z)$, $K_\beta(z)$ denote the modified Bessel functions of order β (Abramowitz and Stegun 1972).

To estimate the total number of bound states (there are no positive-energy bound states by proposition 3) we give the following proposition.

Proposition 2. Let $\gamma > 0$, $g > 0$ and suppose $\int_0^\infty \mathrm{d}r\, [r/(1+r^{1/2})]|V(r)| < \infty$. Then

$$n_l(gV;\gamma;E\leqslant 0) < 2g\int_0^\infty \mathrm{d}r\, rI_{2\lambda-1}[(4\gamma r)^{1/2}]K_{2\lambda-1}[(4\gamma r)^{1/2}]V_-(r). \tag{2.21}$$

Proof. Let $V \leqslant 0$. To include zero-energy bound states we note that

$$(h_l\psi)(r) = 0, \qquad \psi \in D(h_l) \tag{2.22}$$

implies

$$\begin{aligned}\phi(r) &= g\int_0^\infty \mathrm{d}r'\, |V(r)|^{1/2}\hat{g}_l^{(0)}(0,r,r')|V(r')|^{1/2}\phi(r'),\\ \phi(r) &= |V(r)|^{1/2}\psi(r),\end{aligned} \tag{2.23}$$

since $\tilde{\mathscr{F}}_l\,(E=0,g)=0$ in this case. So, following the proof of proposition 1, we obtain (2.21). For general V we note that the infinitesimal form-boundedness of V with respect to $h_l^{(0)}$ implies

$$\dim \mathscr{R}(E_{(-\infty,0]}(h_l)) \leqslant \dim \mathscr{R}(E_{(-\infty,0]}(h_l^-)),$$

where h_l^- denotes the form sum of $h_l^{(0)}$ and $-gV_-$, and $E_{(-\infty,0]}(A)$ represents the non-positive spectral projection of a self-adjoint operator A. Thus

$$n_l(gV;\gamma;E\leqslant 0) \leqslant n_l(-gV_-;\gamma;E\leqslant 0),$$

finishing the proof.

The corresponding short-range result ($\gamma = 0$) reads as follows (Bargmann 1952, Newton 1962).

Corollary 2. Let $\gamma = 0$, $g > 0$ and assume $\int_0^\infty \mathrm{d}r\, r|V(r)| < \infty$. Then

$$n_l(gV;0;E\leqslant 0) < \frac{g}{2\lambda-1}\int_0^\infty \mathrm{d}r\, rV_-(r). \tag{2.24}$$

Remark 1. (a) For a family of optimal bounds for $n_l(gV;0;E\leqslant 0)$, including (2.24) as a special case, see Glaser *et al* (1976, 1978); for a review of other methods compare Simon (1976) and Reed and Simon (1978). A discussion where $V(r)$ is replaced by a nonlocal separable rank-one (Yamaguchi) potential can be found in Van Haeringen *et al* (1977). (b) In the short-range case ($\gamma = 0$) it is well known that $n_l(gV;0;E\leqslant 0)$ increases like $g^{1/2}$ if g tends to infinity (Chadan 1968, Chadan and Mourre 1969, Martin 1977, Grosse 1980). The presence of an additional repulsive Coulomb-type potential $(\alpha^2-\frac{1}{4})/r^2+\gamma/r$, $\gamma>0$ decreases the number of bound states, but in the strong coupling limit $g\to\infty$ this effect should become more and more negligible. In fact, using methods employed in Chadan and Mourre (1969), it is simple to prove

$$\lim_{g\to\infty} g^{-1/2} n_l(gV;\gamma;E\leqslant 0)=\frac{1}{\pi}\int_0^\infty \mathrm{d}r\,|V(r)|^{1/2},\qquad \gamma\geqslant 0, \tag{2.25}$$

if $V(r)\leqslant 0$ and $V(r)\in L^{1/2}(0,\infty)$.

Having discussed the point spectrum $\sigma_p(h_l)$ to some extent, we finally concentrate on the remaining parts of $\sigma(h_l)$.

Proposition 3. For all $l\in N_0$, $\alpha>0$, $\gamma\in R$, $g\in R$, the spectrum of h_l is simple and bounded from below. Its singular continuous part is empty, no positive eigenvalues occur, and the essential spectrum is purely absolutely continuous:

$$\sigma_{ess}(h_l)=\sigma_{ac}(h_l)=[0,\infty).$$

For a proof of proposition 3 compare Weidmann (1967) (cf also Gesztesy *et al* (1980) where a more general result including existence and completeness of various Møller operators is discussed).

Remark 2. (a) For $\gamma>0$, proposition 2 proves the finiteness of $\sigma_p(h_l)$ for potentials $V(r)$ which are, roughly speaking, of order $O(r^{-3/2-\epsilon})$, $\epsilon>0$ as $r\to\infty$. But the explicit structure of h_l shows that there are actually finitely many eigenvalues if $gV(r)\geqslant cr^{-1-\epsilon}$, $\epsilon>0$, $r\geqslant R$ for some $R>0$. The apparent border line $V(r)=O(r^{-3/2-\epsilon})$ as $r\to\infty$ (instead of $V(r)=O(r^{-1-\epsilon})$ as $r\to\infty$) comes from the fact that we used the trace norm $\|\ \|_1$ of the integral operator with kernel $g|V(r)|^{1/2}\hat{g}_l^{(0)}(0,r,r')|V(r')|^{-1/2}V(r')$ in the proof of proposition 2. The class of potentials yielding finitely many eigenvalues is enlarged successively if further norms $\|\ \|_p$, $p=2,3,\ldots$ are taken into account. (b) Most of the results in this section (e.g. (2.16), (2.20), (2.21) if $\gamma>0$, proposition 3) and in the following are also valid for $\alpha=0$ if $D(h_l)$ and the assumptions on $V(r)$ in (2.13)–(2.15) are modified appropriately.

3. Convergence of Born expansions

After introducing the concept of phase shifts $\delta_l(k)$, we derive various lower bounds on the radius of convergence of the Born series for $\tan(\delta_l-\delta_l^{(0)})$, $\exp[2i(\delta_l-\delta_l^{(0)})]$ and related quantities.

Since $E\geqslant 0$ throughout this section we introduce the variable $k=\sqrt{E}$ and redefine F_l and G_l as follows†:

$$F_l^{(0)}(k,\gamma,r)=r^\lambda\,e^{-ikr}{}_1F_1(\lambda-i\gamma/2k;2\lambda;2ikr)=F_l^{(0)}(-k,\gamma,r),\qquad k\geqslant 0, \tag{3.1}$$

$$G_l^{(0)}(k,\gamma,r)=\Gamma(2\lambda)^{-1}\Gamma(\lambda-i\gamma/2k)(2ik)^{2\lambda-1}r^\lambda\,e^{-ikr}\times U(\lambda-i\gamma/2k;2\lambda;2ikr),\qquad k\geqslant 0, \tag{3.2}$$

$$G_l^{(0)}(-k,\gamma,r)=G_l^{(0)}(k,\gamma,r)+2i\frac{B_l(k)}{A_l(k)}F_l^{(0)}(k,\gamma,r)=\overline{G_l^{(0)}(k,\gamma,r)},\qquad k\geqslant 0, \tag{3.3}$$

† In order to simplify the notation we use the same symbols F_l, G_l, $g_l^{(0)}$ etc after E has been replaced by k.

where $A_l(k)$ and $B_l(k)$ are defined by

$$A_l(k)=2^{1-\lambda}k^{-\lambda}\Gamma(2\lambda)|\Gamma(\lambda+i\gamma/2k)|^{-1}\,e^{\pi\gamma/4k},$$
$$B_l(k)=1/kA_l(k)=(2k)^{\lambda-1}\Gamma(2\lambda)^{-1}|\Gamma(\lambda+i\gamma/2k)|e^{-\pi\gamma/4k},\qquad \gamma\in R. \tag{3.4}$$

Then we have

$$F_l(k,\gamma,g,r)=F_l^{(0)}(k,\gamma,r)-g\int_0^r dr'\,g_l^{(0)}(k,\gamma,r,r')V(r')F_l(k,\gamma,g,r'), \tag{3.5}$$

$$F_l(k,\gamma,g,r)=\mathscr{F}_l(-k,\gamma,g)F_l^{(0)}(k,\gamma,r)-g\int_0^\infty dr'\,\hat{g}_l^{(0)}(k,\gamma,r,r')V(r')F_l(k,\gamma,g,r') \tag{3.6}$$

and

$$G_l(k,\gamma,g,r)=G_l^{(0)}(k,\gamma,r)+g\int_r^\infty dr'\,g_l^{(0)}(k,\gamma,r,r')V(r')G_l(k,\gamma,g,r'), \tag{3.7}$$

$$G_l(k,\gamma,g,r)=\mathscr{F}_l(-k,\gamma,g)G_l^{(0)}(k,\gamma,r)-g\int_0^\infty dr'\,\hat{g}_l^{(0)}(k,\gamma,r,r')V(r')G_l(k,\gamma,g,r'), \tag{3.8}$$

where $\mathscr{F}_l(-k,\gamma,g)$ denotes the Wronskian of $G_l(k,\gamma,g,r)$ and $F_l(k,\gamma,g,r)$:

$$\mathscr{F}_l(-k,\gamma,g)=1+g\int_0^\infty dr\,V(r)G_l^{(0)}(k,\gamma,r)F_l(k,\gamma,g,r). \tag{3.9}$$

Insertion of (3.1)–(3.3) into (3.5) then shows

$$F_l(k,\gamma,g,r)=F_l(-k,\gamma,g,r),$$
$$g_l^{(0)}(k,\gamma,r,r')=g_l^{(0)}(-k,\gamma,r,r'),\qquad k\geqslant 0. \tag{3.10}$$

Next we introduce†

$$\exp\{2i[\delta_l(k,\gamma,g)-\delta_l^{(0)}(k,\gamma)]\}=\mathscr{F}_l(-k,\gamma,g)/\mathscr{F}_l(k,\gamma,g),\qquad k>0, \tag{3.11}$$

where

$$\delta_l^{(0)}(k,\gamma)=\arg\Gamma[\tfrac{1}{2}+(l^2+l+\alpha^2)^{1/2}+i\gamma/2k]+\tfrac{1}{2}\pi[l+1/2-(l^2+l+\alpha^2)^{1/2}]$$
$$=\arg\Gamma(\lambda+i\gamma/2k)+\tfrac{1}{2}\pi(l+1-\lambda) \tag{3.12}$$

is the phase shift associated with $h_l^{(0)}$ (cf the Appendix). Since

$$|\mathscr{F}_l(k)-1|=o(1)\qquad \text{as } k\to\infty, \tag{3.13}$$

we choose

$$\delta_l(\infty)=\tfrac{1}{2}\pi(l+1-\lambda)=\tfrac{1}{2}\pi[l+\tfrac{1}{2}-(l^2+l+\alpha^2)^{1/2}] \tag{3.14}$$

in order to guarantee uniqueness of $\delta_l(k)$. (For a detailed discussion of the high-energy behaviour of $\delta_l(k)$ compare Gesztesy *et al* 1980.) With these definitions, the asymptotic

† Note that in general $\mathscr{F}_l(-k,\gamma,g)\neq\overline{\mathscr{F}_l(k,\gamma,g)}$ since $g\in C$.

behaviour of $F_l(k, r)$ and $G_l(k, r)$ reads

$$\begin{aligned}&\left|F_l(k, r)-\mathcal{F}_l(k)\exp\{i[\delta_l(k)-\delta_l^{(0)}(k)]\}A_l(k)\sin\left(kr-\frac{\gamma}{2k}\ln(2kr)-\frac{l\pi}{2}+\delta_l(k)\right)\right|=o(1),\\&\qquad\text{for } k>0,\ r\to\infty,\\&\left|G_l(\pm k, r)-B_l(k)\exp\left[\mp i\left(kr-\frac{\gamma}{2k}\ln(2kr)-\frac{l\pi}{2}+\delta_l^{(0)}(k)\right)\right]\right|=o(1)\\&\qquad\text{for } k>0,\ r\to\infty.\end{aligned}\tag{3.15}$$

We further introduce

$$\tilde{g}_l^{(0)}(k, r, r')=\hat{g}_l^{(0)}(k, r, r')+i\frac{B_l(k)}{A_l(k)}F_l^{(0)}(k, r)F_l^{(0)}(k, r')=\operatorname{Re}\hat{g}_l^{(0)}(k, r, r'),\qquad k\geqslant 0,\tag{3.16}$$

and note that $\tilde{g}_l^{(0)}(k, r, r')$ is real. Then

$$\tilde{F}_l(k, g, r)=\frac{2}{\mathcal{F}_l(k, g)+\mathcal{F}_l(-k, g)}F_l(k, g, r),\qquad k\geqslant 0,\tag{3.17}$$

satisfies

$$\tilde{F}_l(k, g, r)=F_l^{(0)}(k, r)-g\int_0^\infty dr'\,\tilde{g}_l^{(0)}(k, r, r')V(r')\tilde{F}_l(k, g, r'),\qquad k\geqslant 0,\tag{3.18}$$

and from

$$\mathcal{F}_l(k, g)-\mathcal{F}_l(-k, g)=\frac{2ig}{kA_l^2(k)}\int_0^\infty dr\,V(r)F_l^{(0)}(k, r)F_l(k, r),\tag{3.19}$$

one obtains

$$\tan[\delta_l(k, g)-\delta_l^{(0)}(k)]=\frac{-g}{kA_l^2(k)}\int_0^\infty dr\,V(r)F_l^{(0)}(k, r)\tilde{F}_l(k, g, r),\qquad k>0.\tag{3.20}$$

Iterating (3.18) and inserting it into (3.20) then yields the Born expansions (Taylor series in g) for $\tilde{F}_l(k, g, r)$ and $\tan[\delta_l(k, g)-\delta_l^{(0)}(k)]$:

$$\begin{aligned}&\tilde{F}_l(k, g, r)=\sum_{n=0}^{\infty}g^nA_{n,l}(k, r),\\&A_{0,l}(k, r)=F_l^{(0)}(k, r),\end{aligned}\tag{3.21}$$

$$\begin{aligned}A_{n,l}(k, r)=(-1)^n&\int_0^\infty dr_1\,\tilde{g}_l^{(0)}(k, r, r_1)V(r_1)\int_0^\infty dr_2\,\tilde{g}_l^{(0)}(k, r_1, r_2)V(r_2)\dots\\&\times\int_0^\infty dr_n\,\tilde{g}_l^{(0)}(k, r_{n-1}, r_n)V(r_n)F_l^{(0)}(k, r_n),\qquad n=1, 2, \dots,\end{aligned}\tag{3.22}$$

$$\tan[\delta_l(k, g)-\delta_l^{(0)}(k)]=\sum_{n=1}^{\infty}g^nB_{n,l}(k),\tag{3.23}$$

$$B_{n,l}(k)=\frac{-1}{kA_l^2(k)}\int_0^\infty dr\,V(r)F_l^{(0)}(k, r)A_{n-1,l}(k, r),\qquad n=1, 2, \dots.\tag{3.24}$$

From (3.17) and the fact that $\mathscr{F}_l(\pm k, \gamma, g)$ and $F_l(k, \gamma, g, r)$ are entire functions of g, we infer that for fixed k, l, α, γ the radius of convergence for both Born expansions (3.21) and (3.23) is given by the absolute value of that zero of

$$\mathscr{F}_l(k, \gamma, g) + \mathscr{F}_l(-k, \gamma, g) = 0 \tag{3.25}$$

which is closest to the origin $g = 0$. We denote this zero simply by $\tilde{g}_l(k, \gamma)$ (of course it depends on α as well).

In the following we discuss several methods of obtaining a lower bound on the radius of convergence $|\tilde{g}_l(k, \gamma)|$. We first investigate the case $k = 0$ and $\gamma \geqslant 0$.

Proposition 4. Let $k = 0$. Assume

$$\int_0^\infty \mathrm{d}r \frac{r}{1 + r^{1/2}} |V(r)| < \infty \qquad \text{if } \gamma > 0 \qquad \text{and} \int_0^\infty \mathrm{d}r\, r |V(r)| < \infty \qquad \text{if } \gamma = 0.$$

Then

$$|\tilde{g}_l(0, \gamma)| \geqslant \begin{cases} \left[2 \int_0^\infty \mathrm{d}r\, r I_{2\lambda-1}[(4\gamma r)^{1/2}] K_{2\lambda-1}[(4\gamma r)^{1/2}] |V(r)|\right]^{-1}, & \gamma > 0, \\ (2\lambda - 1)\left[\int_0^\infty \mathrm{d}r\, r |V(r)|\right]^{-1}, & \gamma = 0. \end{cases} \tag{3.26}$$

Proof. We first show that $\tilde{g}_l(0, \gamma)$ is necessarily real for $\gamma \geqslant 0$. Since $\mathscr{F}_l(0, \tilde{g}_l) = 0$, the functions $F_l(0, \tilde{g}_l, r)$ and $G_l(0, \tilde{g}_l, r)$ are linearly dependent, which in connection with the estimates (A20), (A22), (A25) and (A27) implies

$$\phi_l(r) = |V(r)|^{1/2} F_l(0, \tilde{g}_l, r) \in L^2(0, \infty)$$

and

$$\|\phi_l\|^2 = \tilde{g}_l(0, \gamma) \int_0^\infty \mathrm{d}r \int_0^\infty \mathrm{d}r'\, \bar{\phi}_l(r) |V(r)|^{1/2} \hat{g}_l^{(0)}(0, r, r') |V(r')|^{1/2} \phi_l(r').$$

From (A10) one infers that $\hat{g}_l^{(0)}(0, \gamma, r, r')$ is real for $\gamma \geqslant 0$ which proves that $\tilde{g}_l(0, \gamma)$ is real for $\gamma \geqslant 0$. From

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{F_l^{(0)}(0, r)}{G_l^{(0)}(0, r)}\right) = \frac{1}{[G_l^{(0)}(0, r)]^2} \geqslant 0 \qquad \text{for all } r \geqslant 0, \gamma \geqslant 0,$$

one shows

$$|\hat{g}_l^{(0)}(0, r, r')| \leqslant F_l^{(0)}(0, r) G_l^{(0)}(0, r') \qquad \text{for all } r, r' > 0, \gamma \geqslant 0. \tag{3.27}$$

Insertion of (3.27) into (3.22) yields ($\tilde{g}_l^{(0)}(0, r, r') = \hat{g}_l^{(0)}(0, r, r')$ if $\gamma \geqslant 0$)

$$|A_{n,l}(0, r)| \leqslant F_l^{(0)}(0, r)\left(\int_0^\infty \mathrm{d}r'\, F_l^{(0)}(0, r') G_l^{(0)}(0, r') |V(r')|\right)^n, \tag{3.28}$$

which implies that

$$|\tilde{g}_l(0, \gamma)| \geqslant \left(\int_0^\infty \mathrm{d}r\, F_l^{(0)}(0, r) G_l^{(0)}(0, r) |V(r)|\right)^{-1}. \tag{3.29}$$

Thus (3.21) converges for any g such that $|g| < [\int_0^\infty \mathrm{d}r\, F_l^{(0)}(0, r) G_l^{(0)}(0, r) |V(r)|]^{-1}$. In the short-range case $\gamma = 0$, (3.26) is due to Jost and Pais (1951) (see also Kohn 1954).

Remark 3. The proof of proposition 4 shows that for $\gamma \geq 0$ the Born series for $\tilde{F}_l(0, g, r)$ at zero energy converges whenever $\pm g V(r)$, $g \in R$ is too weak to support a bound state (or zero-energy resonance). In other words, the expansion (3.21) converges for g in a circle with centre $g = 0$ up to the nearest (real) zero of $\mathscr{F}_l(0, g) = 0$. On the other hand, if one compares the estimate (3.26) with proposition 2 and corollary 2, one infers the weaker statement that the Born series (3.21) is certainly convergent if $-|gV(r)|$ is too weak to support a bound state (for related results in the case $\gamma = 0$ compare Davies 1959/60, Meetz 1962, Huby 1963, Bushell 1972, Amrein *et al* 1977).

Now we turn to the case $k > 0$, $\gamma \in R$.

Proposition 5. Let $k \geq k_0 > 0$, $\gamma \in R$, and $\int_0^\infty \mathrm{d}r\, [r/(1+k_0 r)]|V(r)| < \infty$. Then

$$|\tilde{g}_l(k, \gamma)| \geq \left(\tilde{c}_{\lambda,\gamma}(k_0) \int_0^\infty \mathrm{d}r \frac{r}{1+kr} |V(r)|\right)^{-1}, \tag{3.30}$$

where

$$\tilde{c}_{\lambda,\gamma}(k_0) = \sup_{k \geq k_0} \sup_{r,r'} \left\{ \left(\frac{r}{1+kr}\right)^{-\lambda} \left(\frac{r'}{1+kr'}\right)^{\lambda-1} |\tilde{g}_l^{(0)}(k, \gamma, r, r')| \right\}. \tag{3.31}$$

Proof. With the help of (A16) and (A18) one arrives at

$$|A_{n,l}(k, r)| \leq a_{\lambda,\gamma}(k_0) \left(\frac{r}{1+kr}\right)^{\lambda} \left(\tilde{c}_{\lambda,\gamma}(k_0) \int_0^\infty \mathrm{d}r' \frac{r'}{1+kr'} |V(r')|\right)^{n}, \qquad k \geq k_0 > 0, \tag{3.32}$$

which proves (3.30).

It is intuitively clear that the radius of convergence of the Born expansions (3.21) and (3.23), i.e. $|\tilde{g}_l(k, \gamma)|$, should increase when k becomes larger and larger. This fact is actually confirmed by the following proposition 6, which also indicates how the small-r behaviour of $V(r)$ influences the large-k behaviour of $|\tilde{g}_l(k, \gamma)|$.

Proposition 6. Suppose $\int_0^R \mathrm{d}r\, r^\beta |V(r)| < \infty$ for some $R > 0$ and some $0 \leq \beta \leq 1$. Then

$$|\tilde{g}_l(k, \gamma)| \geq \left[\tilde{c}_{\lambda,\gamma}(k_0) \left(\int_0^R \mathrm{d}r\, r^\beta |V(r)| + k^{-\beta} \int_R^\infty \mathrm{d}r\, |V(r)|\right)\right]^{-1} k^{1-\beta},$$
$$k \geq k_0 > 0, \ \gamma \in R. \tag{3.33}$$

Proof. From proposition 5 we obtain

$$\begin{aligned}
\frac{|\tilde{g}_l(k, \gamma)|}{k^{1-\beta}} &\geq \left(\tilde{c}_{\lambda,\gamma}(k_0) k^{1-\beta} \int_0^\infty \mathrm{d}r \frac{r^{1-\beta}}{1+kr} r^\beta |V(r)|\right)^{-1} \\
&= \left[\tilde{c}_{\lambda,\gamma}(k_0) \left(\int_0^R \mathrm{d}r \frac{(kr)^{1-\beta}}{1+kr} r^\beta |V(r)| + k^{-\beta} \int_R^\infty \mathrm{d}r \frac{kr}{1+kr} |V(r)|\right)\right]^{-1} \\
&\geq \left[\tilde{c}_{\lambda,\gamma}(k_0) \left(\int_0^R \mathrm{d}r\, r^\beta |V(r)| + k^{-\beta} \int_R^\infty \mathrm{d}r\, |V(r)|\right)\right]^{-1}.
\end{aligned}$$

Equation (3.33) shows that even for $\beta = 1$, $\tilde{g}_l(k, \gamma) \to \infty$ as $k \to \infty$ since R may be chosen arbitrarily small.

In the special case $\beta=0$ one can derive an asymptotic formula for $|\tilde{g}_l(k,\gamma)|$ if k tends to infinity (for $\gamma=0$ this has been done by Kohn 1954).

Proposition 7. Assume $\int_0^\infty \mathrm{d}r|V(r)|<\infty$ and $\int_0^\infty \mathrm{d}r\, V(r)\neq 0$. Then

$$|\tilde{g}_l(k,\gamma)|=\frac{\pi k}{|\int_0^\infty \mathrm{d}r\, V(r)|}+\mathrm{o}(k), \qquad \text{as } k\to\infty,\ \gamma\in R. \tag{3.34}$$

Proof. After iterating (3.5) and inserting it into (3.9), one infers

$$\begin{aligned}&\mathscr{F}_l(-k,\gamma,g)=\sum_{n=0}^{\infty} g^n C_{n,l}(k,\gamma),\\ &C_{0,l}(k,\gamma)=1,\end{aligned} \tag{3.35}$$

$$\begin{aligned}C_{n,l}(k,\gamma)=(-1)^{n+1}\int_0^\infty &\mathrm{d}r\, V(r)G_l^{(0)}(k,r)\int_0^r \mathrm{d}r_1\, g_l^{(0)}(k,r,r_1)V(r_1)\\ &\times\int_0^{r_1}\mathrm{d}r_2\, g_l^{(0)}(k,r_1,r_2)V(r_2)\\ &\times\ldots\int_0^{r_{n-2}}\mathrm{d}r_{n-1}\, g_l^{(0)}(k,r_{n-2},r_{n-1})V(r_{n-1})F_l^{(0)}(k,r_{n-1}),\end{aligned} \tag{3.36}$$

$$\begin{aligned}|C_{n,l}(k,\gamma)|\leqslant a_{\lambda,\gamma}(k_0)b_{\lambda,\gamma}(k_0)(c_{\lambda,\gamma}(k_0))^{n-1}k^{-n}(n!)^{-1}\left(\int_0^\infty \mathrm{d}r\frac{kr}{1+kr}|V(r)|\right)^n,\\ k\geqslant k_0>0.\end{aligned} \tag{3.37}$$

With the help of the asymptotic relations (A14) and (A15) and the Riemann–Lebesgue lemma one concludes

$$\lim_{k\to\infty} k^n C_{n,l}(k,\gamma)=\frac{1}{n!}\left(\frac{1}{2\mathrm{i}}\int_0^\infty \mathrm{d}r\, V(r)\right)^n. \tag{3.38}$$

Next we decompose

$$\tfrac{1}{2}[\mathscr{F}_l(k,g)+\mathscr{F}_l(-k,g)]=\cos\left(\frac{g}{2k}\int_0^\infty \mathrm{d}r\, V(r)\right)+R_\lambda(k,g), \tag{3.39}$$

where

$$R_\lambda(k,g)=\sum_{n=0}^{\infty} g^n\,\mathrm{Re}\left[C_{n,l}(k)-\frac{1}{n!}\left(\frac{-1}{2\mathrm{i}k}\int_0^\infty \mathrm{d}r\, V(r)\right)^n\right]. \tag{3.40}$$

If g depends on k such that $|g(k)|\leqslant 2\pi k/|\int_0^\infty \mathrm{d}r\, V(r)|$, then

$$\lim_{k\to\infty}\left|\mathscr{F}_l(\pm k,g(k))-\exp\left(\mp\frac{g(k)}{2\mathrm{i}k}\int_0^\infty \mathrm{d}r\, V(r)\right)\right|=0$$

holds. Thus $|R_\lambda(k,g(k))|<\frac{1}{2}$ for $k\geqslant k_1>0$ and $|g(k)|\leqslant 2\pi k/|\int_0^\infty \mathrm{d}r\, V(r)|$ for k_1 large enough. Thus there exists for all fixed $k\geqslant k_1$ a $\hat{g}_l(k)$ with $0\leqslant|\hat{g}_l(k)|\leqslant 2\pi k/|\int_0^\infty \mathrm{d}r\, V(r)|$ such that $[\mathscr{F}_l(k,\hat{g}_l(k))+\mathscr{F}_l(-k,\hat{g}_l(k))]=0$ for all $k\geqslant k_1$. Since by definition

$$|\tilde{g}_l(k)|\leqslant|\hat{g}_l(k)|\leqslant 2\pi k/|\textstyle\int_0^\infty \mathrm{d}r\, V(r)|,$$

we obtain

$$\lim_{k\to\infty}\left|\tfrac{1}{2}[\mathscr{F}_l(k,\tilde{g}_l(k))+\mathscr{F}_l(-k,\tilde{g}_l(k))]-\cos\left(\frac{\tilde{g}_l(k)}{2k}\int_0^\infty \mathrm{d}r\, V(r)\right)\right|$$

$$=\lim_{k\to\infty}\cos\left(\frac{\tilde{g}_l(k)}{2k}\int_0^\infty \mathrm{d}r\, V(r)\right)=0,$$

which proves

$$|\tilde{g}_l(k)|=\pi k/|\textstyle\int_0^\infty \mathrm{d}r\, V(r)|+\mathrm{o}(k)\qquad \text{as } k\to\infty.$$

Note the close analogy of the high-energy behaviour of $|\tilde{g}_l(k)|$ and $[\delta_l(k,\gamma,g)-\delta_l^{(0)}(k,\gamma)]$ as exhibited in propositions 6 and 7 above and propositions 4 and 5 in Gesztesy *et al* (1980). The additional integrability condition $\int_0^R \mathrm{d}r\, r^\beta|V(r)|<\infty$ leads, for $k\to\infty$, to an increase like $k^{1-\beta}$, $0\leqslant\beta\leqslant 1$ for $|\tilde{g}_l(k)|$, resp. decrease like $k^{\beta-1}$ for $[\delta_l(k,\gamma,g)-\delta_l^{(0)}(k,\gamma)]$.

Now we define

$$\begin{aligned}\tilde{\tilde{g}}_l^{(0)}(k,r,r')&=\hat{g}_l^{(0)}(k,r,r')+2\mathrm{i}\frac{B_l(k)}{A_l(k)}F_l^{(0)}(k,r)F_l^{(0)}(k,r')\\&=\hat{g}_l^{(0)}(-k,r,r'),\qquad k\geqslant 0.\end{aligned}\tag{3.41}$$

Then

$$\tilde{\tilde{F}}_l(k,g,r)=F_l(k,g,r)/\mathscr{F}_l(k,g),\qquad k\geqslant 0,\tag{3.42}$$

satisfies

$$\tilde{\tilde{F}}_l(k,g,r)=F_l^{(0)}(k,r)-g\int_0^\infty \mathrm{d}r'\,\tilde{\tilde{g}}_l^{(0)}(k,r,r')V(r')\tilde{\tilde{F}}_l(k,g,r').\tag{3.43}$$

With the help of (3.19) one obtains

$$\exp\{2\mathrm{i}[\delta_l(k,g)-\delta_l^{(0)}(k)]\}=1-\frac{2\mathrm{i}g}{kA_l^2(k)}\int_0^\infty \mathrm{d}r\, V(r)F_l^{(0)}(k,r)\tilde{\tilde{F}}_l(k,g,r).\tag{3.44}$$

Iterating (3.43) and inserting it into (3.44) then yields the Born expansions for $\tilde{\tilde{F}}_l(k,r)$ and $\exp\{2\mathrm{i}[\delta_l(k)-\delta_l^{(0)}(k)]\}$:

$$\begin{aligned}\tilde{\tilde{F}}_l(k,g,r)&=\sum_{n=0}^\infty g^n D_{n,l}(k,r),\\ D_{0,l}(k,r)&=F_l^{(0)}(k,r),\end{aligned}\tag{3.45}$$

$$\begin{aligned}D_{n,l}(k,r)=(-1)^n&\int_0^\infty \mathrm{d}r_1\,\tilde{\tilde{g}}_l^{(0)}(k,r,r_1)V(r_1)\int_0^\infty \mathrm{d}r_2\,\tilde{\tilde{g}}_l^{(0)}(k,r_1,r_2)V(r_2)\dots\\&\times\int_0^\infty \mathrm{d}r_n\,\tilde{\tilde{g}}_l^{(0)}(k,r_{n-1},r_n)V(r_n)F_l^{(0)}(k,r_n),\qquad n=1,2,\dots,\end{aligned}\tag{3.46}$$

$$\exp\{2\mathrm{i}[\delta_l(k,g)-\delta_l^{(0)}(k)]\}=\sum_{n=0}^\infty g^n E_{n,l}(k),\tag{3.47}$$

$$E_{0,l}(k)=1,$$

$$E_{n,l}(k)=-\frac{2\mathrm{i}g}{kA_l^2(k)}\int_0^\infty \mathrm{d}r\, V(r)F_l^{(0)}(k,r)D_{n,l}(k,r),\qquad n=1,2,\dots.\tag{3.48}$$

From (3.42) it is clear that for fixed k, l, α and γ the radius of convergence for both Born series (3.45) and (3.47) is given by that zero of

$$\mathscr{F}_l(k, \gamma, g) = 0 \tag{3.49}$$

which is closest to the origin $g = 0$. We denote this zero by $\tilde{\tilde{g}}_l(k, \gamma)$.

In the following we briefly discuss several cases where a lower bound for the radius of convergence, i.e. $|\tilde{\tilde{g}}_l(k, \gamma)|$, can be proved.

$k = 0$, $\gamma \geqslant 0$:

Since $\tilde{g}_l(k, \gamma)$ and $\tilde{\tilde{g}}_l(k, \gamma)$ coincide for $k = 0$, $\gamma \geqslant 0$, proposition 4 is valid for $\tilde{\tilde{g}}_l(0, \gamma)$ as well.

$k > 0$, $\gamma \in R$:

Propositions 5 and 6 remain valid for $\tilde{\tilde{g}}(k, \gamma)$ if $\tilde{c}_{\lambda,\gamma}(k_0)$ of (3.31) is replaced by

$$\tilde{\tilde{c}}_{\lambda,\gamma}(k_0) = \sup_{k \geqslant k_0} \sup_{r,r'} \left[\left(\frac{r}{1+kr}\right)^{-\lambda} \left(\frac{r'}{1+kr'}\right)^{\lambda-1} |\tilde{\tilde{g}}_l^{(0)}(k, \gamma, r, r')| \right]. \tag{3.50}$$

Since

$$\tilde{\tilde{g}}_l^{(0)}(k, r, r') = \tilde{g}_l^{(0)}(k, r, r') + \mathrm{i}\frac{B_l(k)}{A_l(k)} F_l^{(0)}(k, r) F_l^{(0)}(k, r')$$

and $\tilde{g}_l^{(0)}(k, r, r')$ is real, we obtain

$$\tilde{\tilde{c}}_{\lambda,\gamma}(k_0) > \tilde{c}_{\lambda,\gamma}(k_0) \qquad \text{for all } k_0 > 0.$$

There is no complete analogue of proposition 7 for $\tilde{\tilde{g}}_l(k, \gamma)$; instead of proposition 7, we now have the following.

Proposition 8. Assume $\int_0^\infty \mathrm{d}r\, |V(r)| < \infty$ and $\int_0^\infty \mathrm{d}r\, V(r) \neq 0$. Then

$$\lim_{k \to \infty} \tilde{\tilde{g}}_l(k, \gamma)/k = \infty, \qquad \gamma \in R. \tag{3.51}$$

Proof. Let $M > 0$. Since

$$\lim_{k \to \infty} \left| \mathscr{F}_l(k, g) - \exp\left(-\frac{g}{2ik} \int_0^\infty \mathrm{d}r\, V(r) \right) \right| = 0 \qquad \text{if } |g| \leqslant Mk,$$

there is a $k_1 > 0$ such that

$$\left| \mathscr{F}_l(k, g) - \exp\left(-\frac{g}{2ik} \int_0^\infty \mathrm{d}r\, V(r) \right) \right| < \tfrac{1}{2}\exp\left(-\tfrac{1}{2}M \left| \int_0^\infty \mathrm{d}r\, V(r) \right| \right)$$

$$\text{for } k \geqslant k_1 \text{ and } |g| \leqslant Mk.$$

Suppose $|\tilde{\tilde{g}}_l(k)| \leqslant Mk$. Then $\mathscr{F}_l(k, \tilde{\tilde{g}}_l(k)) = 0$ implies

$$\exp\left(-\tfrac{1}{2}M \left| \int_0^\infty \mathrm{d}r\, V(r) \right| \right) \leqslant \exp\left(-|\mathrm{Im}\, \tilde{\tilde{g}}_l| \left| \int_0^\infty \mathrm{d}r\, V(r) \right| \Big/ 2k \right)$$

$$\leqslant \exp\left(-\mathrm{Im}\, \tilde{\tilde{g}}_l \int_0^\infty \mathrm{d}r\, V(r)/2k \right) < \tfrac{1}{2}\exp\left(-\tfrac{1}{2}M \left| \int_0^\infty \mathrm{d}r\, V(r) \right| \right),$$

a contradiction. Thus $|\tilde{\tilde{g}}_l(k)|/k > M$ and since M was arbitrary, (3.51) is proved.

This result (in the short-range case $\gamma = 0$ due to Kohn (1954)) shows that in the high-energy limit $\tilde{\tilde{g}}_l(k) > \tilde{g}_l(k)$ if $V(r) \in L^1(0, \infty)$.

Finally we note that it is simple to estimate the truncation error (Kohn 1954, Manning 1965), i.e. the difference between $\tan(\delta_l - \delta_l^{(0)})$ (or $\exp[2i(\delta_l - \delta_l^{(0)})]$) and the first N terms of the associated Born series (3.23) (or (3.47)). For example, using (3.32) we obtain

$$\left|\tan[\delta_l(k, g) - \delta_l^{(0)}(k)] - \sum_{n=1}^{N} g^n B_{n,l}(k)\right| \leqslant \sum_{n=N+1}^{\infty} |g|^n |B_{n,l}(k)|$$

$$\leqslant \frac{|g| a_{\lambda,\gamma}^2(k_0)}{k A_l^2(k)} \int_0^\infty \mathrm{d}r \left(\frac{r}{1+kr}\right)^{2\lambda} |V(r)| \sum_{m=N}^{\infty} \left| g\tilde{c}_{\lambda,\gamma}(k_0) \int_0^\infty \mathrm{d}r' \frac{r'}{1+kr'} |V(r')| \right|^m$$

$$= \frac{|g| a_{\lambda,\gamma}^2(k_0)}{k A_l^2(k)} \int_0^\infty \mathrm{d}r \left(\frac{r}{1+kr}\right)^{2\lambda} |V(r)|$$

$$\times \frac{|g\tilde{c}_{\lambda,\gamma}(k_0) \int_0^\infty \mathrm{d}r' \, [r'/(1+kr')] |V(r')||^N}{1 - |g|\tilde{c}_{\lambda,\gamma}(k_0) \int_0^\infty \mathrm{d}r' \, [r'/(1+kr')] |V(r')|}, \qquad k \geqslant k_0 > 0,$$

and similarly for the other cases.

Appendix

In this Appendix we discuss various properties of $F_l^{(0)}(E, \gamma, r)$, $G_l^{(0)}(E, \gamma, r)$, $F_l(k, \gamma, g, r)$ and $G_l(k, \gamma, g, r)$ (cf §§ 2 and 3 for precise definitions).

Limits ($\lambda = \frac{1}{2} + (l^2 + l + \alpha^2)^{1/2}$)

$$\begin{array}{ccc}
F_l^{(0)}(E, \gamma, r) & \xrightarrow[\alpha \geqslant 0]{E \to 0_-} & \gamma^{1/2-\lambda}\Gamma(2\lambda) r^{1/2} I_{2\lambda-1}[(4\gamma r)^{1/2}] \\
\Big\downarrow {\scriptstyle \alpha \geqslant 0, \ \gamma \to 0_+} & E \leqslant 0, \gamma \geqslant 0 & \Big\downarrow {\scriptstyle \alpha \geqslant 0, \ \gamma \to 0_+} \\
\Gamma(\lambda + \frac{1}{2})(-E/4)^{1/4-\lambda/2} r^{1/2} I_{\lambda-1/2}(\sqrt{-E} r) & \xrightarrow[\alpha \geqslant 0]{E \to 0_-} & r^\lambda
\end{array} \qquad \text{(A1)}$$

$$\begin{array}{ccc}
G_l^{(0)}(E, \gamma, r) & \xrightarrow[\alpha \geqslant 0]{E \to 0_-} & 2\gamma^{\lambda-1/2}\Gamma(2\lambda)^{-1} r^{1/2} K_{2\lambda-1}[(4\gamma r)^{1/2}] \\
\Big\downarrow {\scriptstyle \alpha \geqslant 0, \ \gamma \to 0_+} & E \leqslant 0, \gamma \geqslant 0 & \Big\downarrow {\scriptstyle \alpha > 0, \ \gamma \to 0_+} \\
\Gamma(\lambda + \frac{1}{2})^{-1}(-E/4)^{\lambda/2-1/4} r^{1/2} K_{\lambda-1/2}(\sqrt{-E} r) & \xrightarrow[\alpha > 0]{E \to 0_-} & \dfrac{r^{1-\lambda}}{2\lambda - 1}
\end{array} \qquad \text{(A2)}$$

$$\begin{array}{ccc}
F_l^{(0)}(k, \gamma, r) & \xrightarrow[\alpha \geqslant 0]{k \to 0_+} & \gamma^{1/2-\lambda}\Gamma(2\lambda) r^{1/2} I_{2\lambda-1}[(4\gamma r)^{1/2}] \\
\Big\downarrow {\scriptstyle \alpha \geqslant 0, \ \gamma \to 0_+} & k \geqslant 0, \gamma \geqslant 0 & \Big\downarrow {\scriptstyle \alpha \geqslant 0, \ \gamma \to 0_+} \\
\Gamma(\lambda + \frac{1}{2})(k/2)^{1/2-\lambda} r^{1/2} J_{\lambda-1/2}(kr) & \xrightarrow[\alpha \geqslant 0]{k \to 0_+} & r^\lambda
\end{array} \qquad \text{(A3)}$$

$$
\begin{array}{ccc}
G_l^{(0)}(k,\gamma,r) & \xrightarrow[\alpha\geqslant 0]{k\to 0_+} & 2\gamma^{\lambda-1/2}\Gamma(2\lambda)^{-1}r^{1/2}K_{2\lambda-1}[(4\gamma r)^{1/2}] \\
{\scriptstyle \alpha\geqslant 0}\Big\downarrow{\scriptstyle \gamma\to 0_+} & k\geqslant 0,\ \gamma\geqslant 0 & {\scriptstyle \alpha>0}\Big\downarrow{\scriptstyle \gamma\to 0_+} \\
-\tfrac{1}{2}\mathrm{i}\pi\Gamma(\lambda+\tfrac{1}{2})^{-1}(k/2)^{\lambda-1/2}r^{1/2}H^{(2)}_{\lambda-1/2}(kr) & \xrightarrow[\alpha>0]{k\to 0_+} & \dfrac{r^{1-\lambda}}{2\lambda-1}
\end{array}
\tag{A4}
$$

$$
\begin{array}{ccc}
F_l^{(0)}(k,\gamma,r) & \xrightarrow[\alpha\geqslant 0]{k\to 0_+} & |\gamma|^{1/2-\lambda}\Gamma(2\lambda)r^{1/2}J_{2\lambda-1}[(4|\gamma|r)^{1/2}] \\
{\scriptstyle \alpha\geqslant 0}\Big\downarrow{\scriptstyle \gamma\to 0_-} & k\geqslant 0,\ \gamma\leqslant 0 & {\scriptstyle \alpha\geqslant 0}\Big\downarrow{\scriptstyle \gamma\to 0_-} \\
\Gamma(\lambda+\tfrac{1}{2})(k/2)^{1/2-\lambda}r^{1/2}J_{\lambda-1/2}(kr) & \xrightarrow[\alpha\geqslant 0]{k\to 0_+} & r^{\lambda}
\end{array}
\tag{A5}
$$

$$
\begin{array}{ccc}
G_l^{(0)}(k,\gamma,r) & \xrightarrow[\alpha\geqslant 0]{k\to 0_+} & -\mathrm{i}\pi|\gamma|^{\lambda-1/2}\Gamma(2\lambda)^{-1}r^{1/2}H^{(2)}_{2\lambda-1}[(4|\gamma|r)^{1/2}] \\
{\scriptstyle \alpha\geqslant 0}\Big\downarrow{\scriptstyle \gamma\to 0_-} & k\geqslant 0,\ \gamma\leqslant 0 & {\scriptstyle \alpha>0}\Big\downarrow{\scriptstyle \gamma\to 0_-} \\
-\tfrac{1}{2}\mathrm{i}\pi\Gamma(\lambda+\tfrac{1}{2})^{-1}(k/2)^{\lambda-1/2}r^{1/2}H^{(2)}_{\lambda-1/2}(kr) & \xrightarrow[\alpha>0]{k\to 0_+} & \dfrac{r^{1-\lambda}}{2\lambda-1}
\end{array}
\tag{A6}
$$

Asymptotic behaviour

$$F_l^{(0)}(k,\gamma,r) \underset{r\to\infty}{\overset{k>0}{\sim}} A_l(k)\sin\left(kr-\frac{\gamma}{2k}\ln(2kr)-\frac{l\pi}{2}+\delta_l^{(0)}(k)\right), \qquad \gamma\in R, \tag{A7}$$

$$G_l^{(0)}(\pm k,\gamma,r) \underset{r\to\infty}{\overset{k>0}{\sim}} B_l(k)\exp\left[\mp\mathrm{i}\left(kr-\frac{\gamma}{2k}\ln(2kr)-\frac{l\pi}{2}+\delta_l^{(0)}(k)\right)\right], \qquad \gamma\in R, \tag{A8}$$

where

$$\begin{aligned} A_l(k) &= 2^{1-\lambda}k^{-\lambda}\Gamma(2\lambda)|\Gamma(\lambda+\mathrm{i}\gamma/2k)|^{-1}\,\mathrm{e}^{\pi\gamma/4k}, \\ B_l(k) &= 1/kA_l(k) = (2k)^{\lambda-1}\Gamma(2\lambda)^{-1}|\Gamma(\lambda+\mathrm{i}\gamma/2k)|\,\mathrm{e}^{-\pi\gamma/4k}, \qquad \gamma\in R, \end{aligned} \tag{A9}$$

$$\lim_{k\to 0_+} B_l(k)/A_l(k) = \lim_{k\to 0_+} 1/kA_l^2(k) = \begin{cases} 0, & \gamma\geqslant 0, \\ \pi|\gamma|^{2\lambda-1}\Gamma(2\lambda)^{-2}, & \gamma\leqslant 0, \end{cases} \tag{A10}$$

and

$$\delta_l^{(0)}(k) = \arg\Gamma(\lambda+\mathrm{i}\gamma/2k)+\tfrac{1}{2}\pi(l+1-\lambda). \tag{A11}$$

$$F_l^{(0)}(0,\gamma,r) \underset{r\to\infty}{\sim} \pi^{-1/2}|\gamma|^{1/4-\lambda}\Gamma(2\lambda)r^{1/4}\begin{cases} \tfrac{1}{2}\exp[(4\gamma r)^{1/2}], & \gamma>0, \\ \cos[(4|\gamma|r)^{1/2}-\tfrac{1}{2}\pi(\lambda-\tfrac{1}{2})], & \gamma<0, \end{cases} \tag{A12}$$

$$G_l^{(0)}(0,\gamma,r) \underset{r\to\infty}{\sim} \pi^{1/2}|\gamma|^{\lambda-3/4}\Gamma(2\lambda)^{-1}r^{1/4}\begin{cases} \exp[-(4\gamma r)^{1/2}], & \gamma>0, \\ -\mathrm{i}\exp\{-\mathrm{i}[(4|\gamma|r)^{1/2}-\tfrac{1}{2}\pi(\lambda-\tfrac{1}{2})]\}, & \gamma<0. \end{cases} \tag{A13}$$

$$F_l^{(0)}(k,\gamma,r) \underset{k\to\infty}{\overset{r>0}{\sim}} \pi^{-1/2}\Gamma(\lambda+\tfrac{1}{2})(k/2)^{-\lambda}\sin[kr+\tfrac{1}{2}\pi(1-\lambda)], \qquad \gamma\in R, \tag{A14}$$

$$G_l^{(0)}(k,\gamma,r) \underset{k\to\infty}{\overset{r>0}{\sim}} 2^{-1}\pi^{1/2}\Gamma(\lambda+\tfrac{1}{2})^{-1}(k/2)^{\lambda-1}\exp\{-\mathrm{i}[kr+\tfrac{1}{2}\pi(1-\lambda)]\}, \quad \gamma\in R. \tag{A15}$$

Estimates

$$|g_l^{(0)}(k,\gamma,r,r')| \le c_{\lambda,\gamma}(k_0)\left(\frac{r}{1+kr}\right)^{\lambda}\left(\frac{r'}{1+kr'}\right)^{1-\lambda}, \qquad k\ge k_0>0,\ \gamma\in R, \tag{A16}$$

and analogously for $\hat{g}_l^{(0)}(k,\gamma,r,r')$, $\tilde{g}_l^{(0)}(k,\gamma,r,r')$ and $\tilde{\tilde{g}}_l^{(0)}(k,\gamma,r,r')$; we only have to replace $c_{\lambda,\gamma}(k_0)$ by appropriate constants $\hat{c}_{\lambda,\gamma}(k_0)$, $\tilde{c}_{\lambda,\gamma}(k_0)$ and $\tilde{\tilde{c}}_{\lambda,\gamma}(k_0)$.

$$|g_l^{(0)}(0,\gamma,r,r')| \le c_{\lambda,\gamma}(r,r')^{1/4}\left(\frac{r}{1+r}\right)^{\lambda-1/4}\left(\frac{r'}{1+r'}\right)^{3/4-\lambda}, \qquad \gamma\ne 0; \tag{A17}$$

replacing $c_{\lambda,\gamma}$ by $\hat{c}_{\lambda,\gamma}$, (A17) holds for $\hat{g}_l^{(0)}(0,\gamma,r,r')$ as well. After iterating (3.5) and (3.7) one obtains the estimates

$$\begin{aligned}|F_l(k,\gamma,r)| &\le a_{\lambda,\gamma}(k_0)\left(\frac{r}{1+kr}\right)^{\lambda}\exp\left(c_{\lambda,\gamma}(k_0)|g|\int_0^r \mathrm{d}r'\frac{r'}{1+kr'}|V(r')|\right)\\ &\le \alpha_{\lambda,\gamma}(k_0)\left(\frac{r}{1+kr}\right)^{\lambda}, \qquad k\ge k_0>0,\ \gamma\in R,\end{aligned} \tag{A18}$$

$$\alpha_{\lambda,\gamma}(k_0) = a_{\lambda,\gamma}(k_0)\exp\left(c_{\lambda,\gamma}(k_0)|g|\int_0^\infty \mathrm{d}r'\frac{r'}{1+kr'}|V(r')|\right),$$

$$\begin{aligned}|F_l(k,\gamma,r)-F_l^{(0)}(k,\gamma,r)| &\le c_{\lambda,\gamma}(k_0)\alpha_{\lambda,\gamma}(k_0)\left(\frac{r}{1+kr}\right)^{\lambda}|g|\int_0^r \mathrm{d}r'\frac{r'}{1+kr'}|V(r')|,\\ &k\ge k_0>0,\ \gamma\in R,\end{aligned} \tag{A19}$$

$$\begin{aligned}|F_l(0,\gamma,r)| &\le a_{\lambda,\gamma}r^{1/4}\left(\frac{r}{1+r}\right)^{\lambda-1/4}\exp\left(c_{\lambda,\gamma}|g|\int_0^r \mathrm{d}r'\frac{r'}{1+r'^{1/2}}|V(r')|\right)\\ &\times\begin{cases}\exp(4\gamma r)^{1/2}, & \gamma>0,\\ 1, & \gamma<0,\end{cases}\end{aligned} \tag{A20}$$

$$\begin{aligned}|F_l(0,\gamma,r)-F_l^{(0)}(0,\gamma,r)| &\le c_{\lambda,\gamma}a_{\lambda,\gamma}r^{1/4}\left(\frac{r}{1+r}\right)^{\lambda-1/4}|g|\int_0^r \mathrm{d}r'\frac{r'}{1+r'^{1/2}}|V(r')|\\ &\times\begin{cases}\exp(4\gamma r)^{1/2}, & \gamma>0,\\ 1, & \gamma<0,\end{cases}\end{aligned} \tag{A21}$$

$$|F_l(0,0,r)| \le a_\lambda r^{\lambda}. \tag{A22}$$

$$\begin{aligned}|G_l(k,\gamma,r)| &\le b_{\lambda,\gamma}(k_0)\left(\frac{r}{1+kr}\right)^{1-\lambda}\exp\left(c_{\lambda,\gamma}(k_0)|g|\int_r^\infty \mathrm{d}r'\frac{r'}{1+kr'}|V(r')|\right)\\ &\le \beta_{\lambda,\gamma}(k_0)\left(\frac{r}{1+kr}\right)^{1-\lambda}, \qquad k\ge k_0>0,\ \gamma\in R,\end{aligned} \tag{A23}$$

$$\beta_{\lambda,\gamma}(k_0)=b_{\lambda,\gamma}(k_0)\exp\Big(c_{\lambda,\gamma}(k_0)|g|\int_0^\infty \mathrm{d}r'\,\frac{r'}{1+kr'}|V(r')|\Big),$$

$$|G_l(k,\gamma,r)-G_l^{(0)}(k,\gamma,r)|\leqslant c_{\lambda,\gamma}(k_0)\beta_{\lambda,\gamma}(k_0)\Big(\frac{r}{1+kr}\Big)^{1-\lambda}|g|\int_r^\infty \mathrm{d}r'\,\frac{r'}{1+kr'}|V(r')|,$$

$$k\geqslant k_0>0,\ \gamma\in R, \tag{A24}$$

$$|G_l(0,\gamma,r)|\leqslant b_{\lambda,\gamma}r^{1/4}\Big(\frac{r}{1+r}\Big)^{3/4-\lambda}\exp\Big(c_{\lambda,\gamma}|g|\int_r^\infty \mathrm{d}r'\,\frac{r'}{1+r'^{1/2}}|V(r')|\Big)$$

$$\times\begin{cases}\exp[-(4\gamma r)^{1/2}], & \gamma>0,\\ 1, & \gamma<0,\end{cases} \tag{A25}$$

$$|G_l(0,\gamma,r)-G_l^{(0)}(0,\gamma,r)|\leqslant c_{\lambda,\gamma}b_{\lambda,\gamma}r^{1/4}\Big(\frac{r}{1+r}\Big)^{3/4-\lambda}|g|\int_r^\infty \mathrm{d}r'\,\frac{r'}{1+r'^{1/2}}|V(r')|$$

$$\times\begin{cases}\exp[-(4\gamma r)^{1/2}], & \gamma>0,\\ 1, & \gamma<0,\end{cases} \tag{A26}$$

$$|G_l(0,0,r)|\leqslant b_\lambda r^{1-\lambda}. \tag{A27}$$

Here $c_{\lambda,\gamma}(k_0)$, $c_{\lambda,\gamma}$, $a_{\lambda,\gamma}(k_0),\ldots,b_\lambda$ are appropriate constants.

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